Statistical Measurement of Income Polarization. A cross-national comparison.*

by

Axel Schmidt§
Seminar of Economic and Social Statistics
University of Cologne, Germany

Abstract

While recent research on income polarization is based on only a few approaches, this paper portrays the major methods and applies each to income distributions of Germany (1984-2000) and the US (1984-1997) using the Cross-National Equivalent Files. In addition, statistical inference is provided via bootstrap techniques. Further, we combine kernel density estimation with a unimodality test. The empirical results reveal increasing polarization and inequality in the US while the corresponding figures remain almost constant for Germany.

JEL Codes: C15, D31, D63, I39

Keywords: Income polarization, bootstrap, kernel density estimation, unimodality test

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§**Corresponding address:** Seminar für Wirtschafts- und Sozialstatistik, Universität zu Köln, Albertus-Magnus-Platz, 50923 Köln, Germany. e-mail: schmidt@wiso.uni-koeln.de

1 Introduction

In general, the notion of polarization is captured by describing group formation processes in a society. Polarization deals with building homogeneous clusters that oppose each other. Maximum polarization is reached if half the population is penniless, while the others share the total income equally. It is commonly known that there is a high potential for political unrest in a polarized society. However, the extreme case of total polarization is very unlikely in reality. Nevertheless, research on polarization is mainly motivated by the desire to detect and predict possibilities for social conflict and revolutionary tendencies. Social conflict may develop if ethnic minorities form homogeneous groups deterring other clusters. Differences in race, religion, or social status cause tension if contrasts are sharpened over time. In the following text, we apply the general idea of polarization to income in order to understand the economic reasoning of possible group formations in a society. One important question occurs: "How can one measure income polarization tendencies in a society ?"

The existence of several different approaches to measure income polarization suggests creating a survey of existing methods without favoring one over the others. A special case of income polarization, the 'declining middle class', has been discussed intensively since the early eighties. Researchers have been examining this hypothesis using well-known inequality measures for more than ten years. Esteban and Ray (1994) and Wolfson (1994) independently developed two approaches by forming a new concept of polarization distinct to the measurement of income inequality. Especially Wolfson (1994) provides graphical proof that capturing polarization tendencies using inequality measures is an invalid method. Esteban, Gradín and Ray (1999) improve their previous model and derive Wolfson's measure as a special case for bimodality. Gradín (2000) and D'Ambrosio (2001) suggest further developments of Esteban et al. (1999) by combining the previous model with kernel density estimates.

In addition to the two previous polarization measures and their developments, Jenkins (1995) applies kernel density estimates – a non-parametric technique to estimate the unknown density of e.g., income in an economy – to the UK income distribution and finds growing polarization trends and therefore remarkable evidence in favour of a declining middle class by comparing the income distributions of 1979 and 1990/1991.

In this paper, we provide statistical inference for both polarization measures – Esteban et al. (1999) and Wolfson (1994) – using bootstrap methods. The bootstrap method is very flexible and easily implemented.

In addition, we apply non-parametric kernel density estimates to illustrate income distributions. The chosen bandwidth is a degree of freedom in the estimation of the kernel density. Several methods exist to determine the bandwidth. In general, large values for the window width lead to very smooth densities which tend to be unimodal. In other words, increasing the window width to a certain extent produces unimodal kernel density estimates. In contrast, small values for the window width result in scratchy, multimodal densities.

Instead of choosing one particular method to determine the choice of the window width, we apply a so-called "unimodality test" to obtain statistical inference for kernel density estimates. Starting with small values for the window width, the kernel density estimate is multimodal. By enlarging the window width, the resulting number of modes decreases. This process yields the largest window width for which the kernel density estimate is bimodal. Increasing the window width slightly leads to the optimal window width for which our kernel density estimates are unimodal. Consequently, this particular window width is the smallest value for obtaining unimodal densities. By drawing samples (with replacement) of our income observations and applying kernel density estimates (using the optimal window width), we count the number of modes in these samples and calculate the share of multimodality using this ratio as the underlying p-value of our unimodality test.

In the empirical part of the paper, we apply the two polarization measures and the unimodality test to German and US (unbalanced) panel data using the Cross-National Equivalent Files. The paper is structured as follows: Section two briefly portrays the polarization measure by Esteban, Gradín and Ray. The derivation of Wolfson's measure is presented in section three. Bootstrap methods are explained in section four, while section five gives insights on kernel density estimation and our unimodality test. Section six contains the data analysis while the conclusion is presented in section seven.

2 Income Polarization according to Esteban, Gradín and Ray

Wolfson (1994), and Esteban and Ray (1994) independently developed new concepts of income polarization. While Wolfson's graphical concept is closely related to the Lorenz curve, and consequently the corresponding scalar measure is related to the Gini index, Esteban and Ray derive a class of scalar polarization measures from a set of axioms and theorems. Both Wolfson, and Esteban and Ray state that there exists some type of relationship between their concepts and inequality. Nevertheless, the fundamental Pigou-Dalton axiom of transfers is invalid in their approaches of polarization. Although Esteban and Ray lay cornerstones for measuring polarization, the authors admit drawbacks in their concept and suggested further developments.

Esteban et al. (1999) improve their previous approach and, in addition, derive the scalar Wolfson measure in the special case of bimodality. According to Esteban and Ray (1994) the case of bipolarization features the greatest level of income polarization. Therefore, Wolfson's measure represents an extreme value of a general class of polarization measures provided by Esteban et al. (1999).

While Esteban and Ray (1994) assume pre-grouped income data, Esteban et al. (1999) admit that the assumption of pre-grouped data might not fit with subjective identification and alienation of individuals as proposed in their earlier measure. By accounting for an error term, caused by regrouping the pregrouped data according to both identification and alienation functions, the new polarization measure is applicable to all different kinds of income distributions.

In particular, Esteban and Ray state one main difference between inequality and income polarization. While income inequality is mainly based on the Pigou-Dalton principle of transfers on which the tendency of inequality can be derived independently from the underlying income distribution, income polarization considers the whole income distribution. Therefore, Esteban and Ray label income inequality as a 'local' and income polarization as a 'global' concept.

In the following, we briefly portray Esteban et al. (1999). Their behavioral model is based on identification and alienation functions. The identification function $I(\cdot)$ accounts for intra-group homogeneity indicating that individuals feel related to others in the same income group. Further, the alienation function $a(\cdot)$ comprises inter-group heterogeneity whereby individuals in different income groups reject each other. Individuals consider other incomes within a range D as part of their own group. In fact, a lack of alienation, $|y_i - y_j| \leq D \ \forall i, j$, leads to the existence of only one group with nil polarization, $P = 0$.

Esteban and Ray (ER) define income polarization as the sum of all effective antagonisms which are the connection of intra-group and inter-group sentiments:

$$P^{ER}(\boldsymbol{\pi}, \mathbf{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \pi_j T\left(I\left(\pi_i\right), a\left(|y_i - y_j|\right)\right).$$

Note that effective antagonism $T(\cdot)$ is assumed to be additive and depends, besides on a vector of population weights $\boldsymbol{\pi}$, on a vector of properties, e.g., income $\mathbf{y}$.

After applying several axioms, Esteban and Ray present a general polarization measure

$$P^{ER}(\pi, \mathbf{y}) = K \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j |y_i - y_j| \tag{1}$$

with K as a constant for population normalization. In addition, the authors derive the 'polarization sensitivity' $\alpha \in [1, 1.6]$ using numerical computation.

Nevertheless, the identification function $I(\pi) = \pi^{\alpha}$ with the polarization sensitivity α separates the concepts of inequality and polarization. In fact, ignoring identification, $\alpha = 0$, Esteban and Ray's polarization measure P^{ER} is proportional to the Gini index. Consequently, income polarization antagonizes inequality for increasing polarization sensitivity α.

By extending their former measure using a statistical approach, Esteban et al. (1999) made redundant the pre-grouped income distribution. By regrouping the pre-grouped society according to identification and alienation functions, the polarization measure is biased. Therefore, the authors extend the former polarization measure in equation (1), but correct it by an error term.

In order to initiate the error term extension of the polarization measure, Esteban et. al. use the following n-group representation ρ of an income distribution F with ascending ordered incomes, $y_0 = a < \ldots < y_n = b$:

$$(y_0, y_1, \ldots, y_n; \pi_1, \ldots, \pi_n; \mu_1, \ldots, \mu_n).$$

Furthermore, Esteban et. al. define

$$\begin{aligned} \pi_i &= \int_{y_{i-1}}^{y_i} f(y)\,\mathrm{d}y \\ \mu_i &= \frac{1}{\pi_i} \int_{y_{i-1}}^{y_i} y f(y)\,\mathrm{d}y \end{aligned}$$

for all $i = 1, \ldots, n$. π_i indicates the probability that an income y resides in the defined income class i, whereas μ_i reflects a weighted average income in the particular income class i.

Unless the number of groups tends to the number of individuals, $n \to N$, respectively households, each n-group representation is inaccurate as an approximation of the income distribution F, and has consequently been corrected by an error term $\varepsilon(f, \rho)$. Therefore, Esteban et al. (1999) (EGR) define the polarization measure as

$$\begin{aligned} P^{EGR}(f, \alpha, \beta) &= ER(\alpha, \rho) - \beta\varepsilon(f, \rho) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j \left|\mu_i - \mu_j\right| - \beta\varepsilon(f, \rho) \end{aligned} \tag{2}$$

where β is a weight indicating the amount of restructuring which the error term corrects.

While Esteban et. al. consider the number of groups, n, of a density representation as given, much attention is drawn to the question about limits of particular income classes. In order to answer the question about the particular location of the groups, the authors measure the error term $\varepsilon(\cdot)$ in the following way

$$\varepsilon(f,\rho)=\frac{1}{2}\sum_i\int_{y_{i-1}}^{y_i}\int_{y_{i-1}}^{y_i}|x-z|\,f(x)\,f(z)\,\mathrm{d}x\,\mathrm{d}z.$$

For reducing the approximation error $\varepsilon(f,\rho)$, an accurate choice of the representation ρ is important. Recognizing that the particular representation error in each income group i is measured by the Gini index, the average distance between incomes within each group is therefore minimized in order to derive a maximum degree of homogeneity within each group.

Assuming ρ^* as the particular n-group representation that minimizes the in-group dispersion, Esteban et. al. express the solution of the previous optimization as

$$y_i^*=\frac{\pi_i^*\mu_i^*+\pi_{i+1}^*\mu_{i+1}^*}{\pi_i^*+\pi_{i+1}^*}.$$

Therefore, the minimization indicates the weighted average income of two adjoining income groups. In other words, the Lorenz curve is approximated by linear functions by decomposing the Lorenz curve into n parts. For $n\rightarrow N$, these approximations converge to the Lorenz curve. Consequently, the error term $\varepsilon(\cdot)$ can be stated as

$$\varepsilon(f,\rho^*)=G(f)-G(\rho^*) \tag{3}$$

where G is the Gini index. Finally, consolidating equations (2) and (3), Esteban, Gradín and Ray derive the following polarization measure

$$\begin{aligned}P^{EGR}(f,\alpha,\beta) &= ER(\alpha,\beta)-\beta\left[G(f)-G(\rho^*)\right]\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i^{1+\alpha}\pi_j\left|\mu_i-\mu_j\right|-\beta\left[G(f)-G(\rho^*)\right]. \qquad (4)\end{aligned}$$

In contrast to all existing studies based on Esteban et al. (1999), we prefer not to use the logarithm of the variable 'income' due to lower comparability to both the Gini index and Wolfson's polarization measure.

3 The Special Case of Bimodality and the Derivation of Wolfson's Polarization Measure

Esteban et al. (1999) derive Wolfson's polarization measure in the special case of a bimodal income distribution. Wolfson's polarization measure (according to Wolfson (1994)) is defined as

$$P^W := 2\frac{\mu}{m}\left(1 - 2L\left(0.5\right) - G\right),$$

where μ stands for the mean, m for the median, $L\left(0.5\right)$ for the Lorenz curve at the median, and G for the Gini index. Therefore, the following implementations are based on the special case of a 2-group representation of an income distribution. Bipolarization is, according to Esteban and Ray (1994), the case featuring the highest level of income polarization. Consequently, the authors identify the probability of the demarcation income y between these two opposing groups as

$$\pi = \int_a^y f\left(x\right) \mathrm{d}x.$$

In addition, they assume the underlying income distribution as mean normalized, $\mu = 1$. Furthermore, let the Lorenz curve be represented by $L\left(\cdot\right)$. Consequently, the particular mean income of each group is

$$\mu_1 = \frac{L\left(\pi\right)}{\pi}$$

and

$$\mu_2 = \frac{1 - L\left(\pi\right)}{1 - \pi}.$$

Therefore, the former polarization measure (see equation (4)), transforms in the case of a 2-group representation to

$$\begin{aligned} P^{EGR}\left(\alpha, \rho\right) &= \left(\pi^{1+\alpha}\left(1-\pi\right) + \left(1-\pi\right)^{1+\alpha}\pi\right)\left(\mu_2 - \mu_1\right) - \beta\varepsilon\left(f, \rho\right) \\ &= \left(\pi^\alpha + \left(1-\pi\right)^\alpha\right)\left(\pi - L\left(\pi\right)\right) - \beta\varepsilon\left(f, \rho\right). \end{aligned}$$

As a result, the error term develops to

$$\varepsilon\left(f, \rho\right) = G - \left(\pi - L\left(\pi\right)\right).$$

Therefore, without minimizing the area between the Lorenz curve and the linear approximation, Esteban et al. (1999) state the polarization measure as

$$P^{EGR}\left(f; \alpha, \beta, y\right) = \left(\pi^\alpha + \left(1-\pi\right)^\alpha\right)\left(\pi - L\left(\pi\right)\right) - \beta\left(G - \left(\pi - L\left(\pi\right)\right)\right). \quad (5)$$

Minimization yields the cut-off income which is equal to the median income, $y = m$. Consequently, the probability for incomes to reside in the first group is $\pi = \frac{1}{2}$ which leads to Wolfson's polarization measure in the special case of $\alpha = \beta = 1$:

$$\begin{aligned} P^{EGR}(f;1,1,y) &= \left(2\cdot\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}-L\left(\frac{1}{2}\right)\right)-\left(G-\left(\frac{1}{2}-L\left(\frac{1}{2}\right)\right)\right) \\ &= 2\cdot\left(\frac{1}{2}-L\left(\frac{1}{2}\right)\right)-G \\ &= \frac{m}{2}P^{W}\,. \end{aligned}$$

4 Bootstrap Statistical Inference

In this paper, we use bootstrap methods to obtain statistical inference for polarization measures. Related studies on the bootstrap are for instance Biewen (2002), and Mills and Zandvakili (1997). Biewen applies bootstrap techniques for measures of inequality, mobility and poverty. In addition, the author shows the consistency of bootstrap results for those measures specifically taking decompositions by subgroups and income source, and forms of intra-household correlation into account. In contrast, Mills and Zandvakili provide confidence intervals and hypothesis tests for several inequality indices without stressing any theoretical framework.

Statistical inference using bootstrap methods is done by taking samples from the sample. By repeating this sampling procedure many times, one gets intervals for means, variances, confidence intervals, and many other statistical figures. Therefore, bootstrapping is easily implemented and in addition, it is a very flexible tool. Dealing with dependencies, sampling weights, or other special statistical properties is one major advantage of bootstrapping over usual inference methods for which the resulting formulas are strongly mathematical, often complicated and in some cases even not derivable at all. The δ-method is one important method for variance estimation. After expressing a particular measure as a function of moments, the δ-method produces the resulting estimate for the variance of this measure. Such as, for instance, Wolfson's polarization measure P^W is a function of the Gini index which is inexpressible as a function of moments. Therefore, the δ-method is inapplicable to the majority of income polarization measures. Consequently, we use bootstrap methods to obtain statistical inference for all presented polarization measures.

5 Kernel Density Estimation and a Unimodality test

Kernel density estimation is a method to estimate the unknown density function for e.g., income. It omits both the problem of arbitrary income interval framing and discounts the problematic classification into categories for histograms.

The estimate for the density function at x is

$$\hat{f}(x;h) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)$$

whereby n denotes the total number of observations, h the bandwidth of the window, x_i the income of the i^{th} individual, $K(\cdot)$ the kernel function, and $\hat{f}(x, h)$ the estimated density for the particular income at x. In general, the kernel function possesses similar properties to a density function. Therefore, a widely used kernel function $K(\cdot)$ is the standard normal distribution $\Phi(\cdot)$.

Schluter (1998) examines income dynamics applying mobility measures, stochastic kernel densities, and among other approaches, a unimodality test based on Silverman (1981) using the PSID-GSOEP 'Equivalent Datafile', a predecessor of the Cross-National Equivalent Files.

The following briefly applies bootstrap methods to a unimodality test connected with kernel density estimates, and is based on Efron and Tibshirani (1993). The unimodality test yields the smallest window width h_{opt} for which the resulting kernel density is unimodal. Starting with small values for the window width, the kernel density estimate is multimodal. By enlarging the window width, the resulting number of modes decreases. This process yields the largest window width h for which the kernel density estimate is bimodal. Incrementing the window width slightly leads to the optimal window width h_{opt} for which our kernel density estimates are unimodal. Consequently, the window width h_{opt} is the smallest value for obtaining unimodal densities. In addition, this particular window width h_{opt} is our test statistic and equals the critical value for testing unimodality. For further details, see Appendix B.

6 Data Analysis

The Cross-National Equivalent Files consists of unit-record data variables. We equalize household income according to the OECD equivalent scale with an elasticity scale factor of 0.5 to account for economies of scale of household members compared to solitarily living individuals. In addition, the household income is deflated using the consumer price index for Germany, respectively for the US. For more information on the data, see Appendix A.

The following data analysis, using the Cross-National Equivalent Files, reveals different pictures regarding the polarization process in Germany and the US. In general, our results show increasing polarization and the often cited, growing inequality in the US while the corresponding figures remain almost constant for Germany. The analysis of the German data has to be divided into two parts due to the political union with the former German Democratic Republic in 1990. As a result the corresponding measures are biased in the years 1990 and 1991 and are therefore left unconsidered in the following analysis. In addition, a new sample F was included in the year 2000 which also leads to biased results in the corresponding figures and are left unconsidered as well. While Wolfson's polarization measure slightly increases by an average increase of 1.08% per year for Germany (until 1989), inequality stays constant until 1989, grows afterwards and features a slight decline in the late nineties. Therefore, inequality shows an average increase of 0.42% per year and Wolfson's polarization index grows by 0.82% on average per year until 1999. While Esteban, Gradín and Ray's polarization measure shows a steady average growth rate of 2.21% per year for four groups and $\alpha = 1.6$, polarization in the case of two groups ($\alpha = 1$) seems to increase slightly by 0.75% per year.

In general, kernel density estimates vary in both form and shape according to different horizontal axes. In order to provide useful estimates for the unimodality test, especially upper income ranges need adjustment. Specifically adapted kernel density estimates are specialized to adjust the underlying bandwidth in upper income ranges. Nevertheless, our unimodality test assumes a fixed bandwidth for the kernel estimates in order to determine the critical bandwidth as a test statistic. Consequently, higher incomes disturb the shape of the kernel estimates and are therefore unconsidered according to a rule which first calculates the differences of the logarithm of the income, and secondly, starting at the highest income, the program censors all incomes above the particular income, for which the ten previous income differences were below 5%. In addition, Kernel density estimates show little, single bumps in upper income ranges which lead to an oversmoothing in the unimodality test routine. Consequently, a reasonable unimodality test takes only those modes into account which are above an arbitrarily defined horizontal demarcation line. Therefore, we carried out a sensitivity analysis in order to find a critical horizontal demarcation line for accepting modes. By increasing the height of the horizontal demarcation line, the critical bandwidth decreases. As a tolerance line equal to the abscissa leads to an oversmooth kernel density estimate, we begin with a tolerance level of 2.5% and choose the highest horizontal demarcation line where the critical bandwidth remains constant. We reject unimodality for the majority of post-governmental income for Germany. Unfortunately, our unimodality test cannot provide further inside in the current polarization situation.

The picture in the US is totally different from the findings for Germany. A new way of interviewing (computer assisted telephone interviewing, CATI) was started in 1993 which leads to biased measures since 1994. Consequently, the following analysis left the last four years unconsidered. While the Gini index shows increasing inequality, Wolfson's, and Esteban, Gradín and Ray's polarization measures feature growing polarization. The Gini index and Wolfson's polarization measure has values of 1.91% and 1.83% average growth per year for post-tax/post-transfer income (until 1993). In addition to Wolfson's polarization measure, Esteban, Gradín and Ray's polarization measure increases by 2.36% and 0.92% for two ($\alpha = 1$), respectively four groups ($\alpha = 1.6$) for post-government transfers on average per year (until 1993). The unimodality test gives no further insight into the US income distribution. In fact, we reject unimodality in 50% of all years.

To conclude, inequality and polarization constantly rises in the US for post-tax/post-transfer income, while the corresponding measures remain almost constant for Germany during the years. The obtained standard errors of our inequality and polarization measures deserve closer attention. While all standard errors for the Gini inequality measure and Wolfson's polarization measure are quite accurate for the US and Germany, Esteban, Gradín and Ray's polarization measure results only accurate standard errors for German income data. In case of the US, the corresponding standard errors are unacceptable. Even increasing the number of bootstrap repetitions cannot solve the problem for US income data. The major source of inaccuracy is the linear approximation of the Gini index in the error correction term of EGR polarization measure. One possible explanation for this dilemma might be the higher level of inequality and polarization in the US income distribution (compared to German income data) which results more often in 'unfavorable' samples. In addition, our unimodality test cannot provide further insight in both income distributions. While this test offers excellent theoretical properties the practical application often leads to ambiguous results.

7 Conclusion

In this paper, we portray all major income polarization measures and apply them using the Cross-National Equivalent Files from 1984-2000 for Germany and from 1984-1997 for the US.

While previous research in this field is mainly based on Esteban et al. (1999), which is a successor of Esteban and Ray (1994), Wolfson's polarization concept has hardly been examined in recent studies. Esteban, Gradín and Ray's polarization measure is more general and includes Wolfson's measure as a special case for two groups.

In addition to the theoretical presentation of polarization measures, we apply kernel density estimates as a non-parametric method to estimate the density of the underlying income distributions. Besides several assets of kernel density estimates, one drawback is the lack of statistical significance of possible findings. Therefore, we combine kernel density estimates with a unimodality test to obtain statistically significant conclusions for any kind of findings.

Besides obtaining statistical inference for all measures and the kernel density estimates, we find growing inequality and polarization in the US income (after tax and government transfers). In contrast, the results for Germany reveal a constant level of inequality and polarization. In comparison to the US, Germany has a lower level of income inequality and polarization.

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A Data Description: The Cross-National Equivalent Files

The Department of Policy Analysis and Management at Cornell University in conjunction with the 'German Institute for Economic Research' (DIW Berlin) and the University of Michigan have created the Cross-National Equivalent Files (CNEF), merging major panel data sets, among others, the United States Panel Study of Income Dynamics (PSID) and the German Socio-Economic Panel (GSOEP), to facilitate cross-national research.

The Cross-National Equivalent Files features various advantages over the independent data files. While both single data sets have grown in size and complexity during the years, the CNEF represents a simplified version and is especially qualified for comparative research projects between different countries. The simplicity of the data structure results in low start-up costs for beginners in empirical data analysis and experienced researchers alike. In contrast to descriptive books on the structure of the PSID and the GSOEP, the description of the CNEF is brief.

The data has been researched and structured with the focus on solving practical research problems. As the data collection methods vary for the PSID and GSOEP, the concept of 'income' differs for the single data sets. In contrast, the CNEF consists, among other variables, of constructed variables which are not included in the original data sets like 'household post-government income' for obtaining post-tax/post-transfer household income. Furthermore, a reliability code is provided specifying the level of comparability of the particular cross-national variables wherein '1' represents complete comparability and '4' indicates no comparability at all.

The current family composition in both data sets are either original sample members or their offspring, which illustrates a dynamic family development situation and independently generates new sample members. While the PSID interviews only the family head regarding questions about other family members, the GSOEP questions all family members older than 16 years. For further details on the CNEF, see Burkhauser, Butrica, Daly and Lillard (2001).

The following presents core facts about both independent data sets. The GSOEP is a longitudinal data set consisting data on German households and individuals and was founded in 1983 by the Special Research Area 3 'Microanalytical Basis of Social Politics' (Sfb 3) at the Universities of Frankfurt and Mannheim during the years 1979-1982. Nowadays, the DIW Berlin organizes and maintains this very complex data set.

In the very beginning, the GSOEP contained almost 6,000 households and more than 12,000 individuals. Since 1984, the sample has besides German households, also included Turkish, Jugoslavian, Greek, Italian, and Spanish households which are overrepresented in the sample. Therefore, sampling weights adjust for the disproportionate number of non-German households in the sample. For further details about the GSOEP, see Hanefeld (1987) or SOEP Group (2001).

The Panel Study of Income Dynamics is one of the most significant and influential data sets in social sciences. It was founded in 1968 for poverty research studies and is conducted by the Survey Research Center at the University of Michigan. The longitudinal survey of US households and individuals consists of many different dynamic aspects including economic, demographic, sociological, and psychological facets. The data set has grown from 4,802 households and 18,000 individuals in 1968 to about 9,829 households and 53,013 individuals in 1997. One drawback is that households with low incomes are overrepresented, which leads to a disproportionately large subsample of non-white households. It is counterbalanced through 'probability-of-selection' weights in order to represent the US population. For further details about the PSID, see Hill (1992).

B Statistical Method: Kernel Density Estimation and a Unimodality Test

The following portrays the unimodality test according to Efron and Tibshirani (1993). As stated above, smaller window widths lead to multimodality. Instead of repeatedly drawing samples, obtaining the optimal window width for each sample, and checking whether the resulting window width h_{opt} is greater or smaller than our window width h_{opt}, we equivalently simply count the number of modes in the kernel density estimate for each sample.

Therefore, the underlying test statistic is

$$\begin{aligned} H_0 &: \# \text{ of modes} = 1 \\ H_1 &: \# \text{ of modes} > 1 \end{aligned}$$

using the null hypothesis that the kernel density has at most one mode. The alternative hypothesis equals checking whether the obtained window width of the last sample is significantly smaller than h_{opt}. We reject the null hypothesis if the sample window width is too small compared to the optimal window width h_{opt}. In the same manner, we particularly check if the number of modes in the kernel density estimate is greater than one which indicates multimodality.

How does the unimodality test work? By drawing samples with replacement from the kernel density estimate $\hat{Y}_{opt}$, which has been created with the optimal window width $\widehat{h}_{opt} := const.$, and counting the number of modes in these samples, we calculate the share of multimodality which is in addition, the underlying p-value of our unimodality test.

Unfortunately, drawing pseudo random samples with replacement from our original sample and adding independently and identically distributed noise, ϵ_i, afterwards leads to

$$r_i = y_i^* + \widehat{h}_{opt}\epsilon_i$$

where $\widehat{h}_{opt}\epsilon_i \sim N\left(0, \widehat{h}^2_{opt}\right)$. Therefore, our kernel density develops to

$$\begin{aligned}
\hat{f}\left(r_i; h_{opt}\right) &= \frac{1}{n\widehat{h}_{opt}} \sum_{i=1}^{n} \Phi\left(\frac{r_i - x_i}{\widehat{h}_{opt}}\right) \\
&= \frac{1}{n\widehat{h}_{opt}} \sum_{i=1}^{n} \Phi\left(\frac{y_i^* + \widehat{h}_{opt}\epsilon_i - x_i}{\widehat{h}_{opt}}\right) .
\end{aligned}$$

As a result, we get

$$\begin{aligned}
E\left(r_i\right) &= E\left(y_i^*\right) + \widehat{h}_{opt} E\left(\epsilon_i\right) \\
&= E\left(y_i^*\right) = \overline{y}^* ,
\end{aligned}$$

$$\begin{aligned}
Var\left(r_i\right) &= Var\left(y_i^*\right) + Var\left(\widehat{h}_{opt}\epsilon_i\right) \\
&= Var\left(y_i^*\right) + \widehat{h}^2_{opt} Var\left(\epsilon_i\right) \\
&= \widehat{\sigma}^2 + \widehat{h}^2_{opt}
\end{aligned}$$

for the expected value and the variance. In particular, the variance of r_i is greater than the plug-in variance $\widehat{\sigma}^2$. Therefore, we correct the bias by multiplying with

$$\frac{1}{\sqrt{1 + \frac{\widehat{h}^2_{opt}}{\widehat{\sigma}^2}}} .$$

Instead of using r_i as input for the kernel density estimate, we use

$$x_i^* = \overline{y}^* + \frac{1}{\sqrt{1 + \frac{\widehat{h}^2_{opt}}{\widehat{\sigma}^2}}} \left[y_i^* - \overline{y}^* + \widehat{h}_{opt}\epsilon_i\right]$$

which leads to a consistent and unbiased estimator:

$$
\begin{aligned}
E\left(x_i^*\right) &= E\left(\overline{y}^*\right) + \frac{1}{\sqrt{1+\dfrac{\widehat{h}_{opt}^2}{\widehat{\sigma}^2}}}\left[E\left(y_i^*\right) - E\left(\overline{y}^*\right) + \widehat{h}_{opt} E\left(\epsilon_i\right)\right] \\
&= \overline{y}^* + \frac{1}{\sqrt{1+\dfrac{\widehat{h}_{opt}^2}{\widehat{\sigma}^2}}}\left[\overline{y}^* - \overline{y}^*\right] \\
&= \overline{y}^* = E\left(r_i\right).
\end{aligned}
$$

$$
\begin{aligned}
Var\left(x_i^*\right) &= \frac{1}{1+\dfrac{\widehat{h}_{opt}^2}{\widehat{\sigma}^2}}\left[Var\left(y_i^*\right) + \widehat{h}_{opt}^2\right] \\
&= \frac{\widehat{\sigma}^2 + \widehat{h}_{opt}^2}{\widehat{\sigma}^2 + \widehat{h}_{opt}^2}\,\widehat{\sigma}^2 \\
&= \widehat{\sigma}^2.
\end{aligned}
$$

C Tables

Germany	G	P^W	$P^{EGR}\,(\alpha=1)$			$P^{EGR}\,(\alpha=1.6)$		
Year			$n=2$	$n=3$	$n=4$	$n=2$	$n=3$	$n=4$
1984	0.2513	0.1880	0.0947	0.1044	0.0911	0.0377	0.0383	0.0289
	(0.0043)	(0.0036)	(0.0017)	(0.0017)	(0.0018)	(0.0094)	(0.0015)	(0.0010)
1985	0.2454	0.1985	0.0969	0.1039	0.0899	0.0394	0.0381	0.0283
	(0.0043)	(0.0028)	(0.0013)	(0.0016)	(0.0016)	(0.0010)	(0.0011)	(0.0013)
1986	0.2851	0.2072	0.1031	0.1181	0.1035	0.0398	0.0452	0.0341
	(0.0063)	(0.0035)	(0.0019)	(0.0027)	(0.0028)	(0.0012)	(0.0020)	(0.0019)
1987	0.2493	0.1997	0.0981	0.1037	0.0909	0.0403	0.0376	0.0289
	(0.0040)	(0.0030)	(0.0015)	(0.0016)	(0.0019)	(0.0010)	(0.0014)	(0.0017)
1988	0.2524	0.2026	0.0999	0.1057	0.0925	0.0413	0.0383	0.0294
	(0.0032)	(0.0033)	(0.0026)	(0.0015)	(0.0015)	(0.0009)	(0.0011)	(0.0014)
1989	0.2517	0.1984	0.0980	0.1043	0.0917	0.0401	0.0371	0.0290
	(0.0050)	(0.0032)	(0.0054)	(0.0015)	(0.0018)	(0.0011)	(0.0011)	(0.0015)
1990	0.4007	0.3055	0.1502	0.1760	0.1577	0.0566	0.0678	0.0561
	(0.0052)	(0.0046)	(0.0080)	(0.0030)	(0.0029)	(0.0070)	(0.0016)	(0.0019)
1991	0.3993	0.3132	0.1552	0.1768	0.1569	0.0609	0.0678	0.0545
	(0.0042)	(0.0048)	(0.0132)	(0.0030)	(0.0026)	(0.0041)	(0.0016)	(0.0024)
1992	0.2635	0.2190	0.1071	0.1117	0.0961	0.0459	0.0408	0.0299
	(0.0028)	(0.0026)	(0.0057)	(0.0016)	(0.0016)	(0.0013)	(0.0019)	(0.0022)
1993	0.2646	0.2187	0.1070	0.1123	0.0968	0.0451	0.0414	0.0310
	(0.0027)	(0.0026)	(0.0043)	(0.0014)	(0.0021)	(0.0030)	(0.0024)	(0.0022)
1994	0.2668	0.2200	0.1085	0.1117	0.0997	0.0467	0.0406	0.0377
	(0.0031)	(0.0030)	(0.0184)	(0.0029)	(0.0028)	(0.0122)	(0.0035)	(0.0043)
1995	0.2702	0.2165	0.1065	0.1136	0.1019	0.0420	0.0417	0.0359
	(0.0033)	(0.0029)	(0.0221)	(0.0036)	(0.0034)	(0.0114)	(0.0030)	(0.0036)
1996	0.2705	0.2149	0.1061	0.1123	0.1027	0.0438	0.0410	0.0320
	(0.0031)	(0.0028)	(0.0218)	(0.0048)	(0.0028)	(0.0148)	(0.0033)	(0.0036)
1997	0.2701	0.2125	0.1056	0.1120	0.1025	0.0438	0.0442	0.0324
	(0.0036)	(0.0033)	(0.0233)	(0.0180)	(0.0032)	(0.0124)	(0.0032)	(0.0034)
1998	0.2693	0.2156	0.1067	0.1116	0.0985	0.0449	0.0454	0.0371
	(0.0038)	(0.0033)	(0.0192)	(0.0074)	(0.0029)	(0.0194)	(0.0031)	(0.0044)
1999	0.2678	0.2125	0.1060	0.1115	0.0982	0.0447	0.0395	0.0401
	(0.0038)	(0.0035)	(0.0191)	(0.0043)	(0.0031)	(0.0123)	(0.0029)	(0.0037)
2000	0.3303	0.2215	0.1229	0.1370	0.1249	0.0536	0.0522	0.0495
	(0.0195)	(0.0039)	(0.0146)	(0.0108)	(0.0094)	(0.0079)	(0.0099)	(0.0081)

Table 1: Inequality and Polarization Indices for Germany 1984-2000 (Standard errors in parenthesis; obtained using 200 bootstrap repetitions)

US	G	P^W	$P^{EGR}\,(\alpha=1)$			$P^{EGR}\,(\alpha=1.6)$		
Year			$n=2$	$n=3$	$n=4$	$n=2$	$n=3$	$n=4$
1984	0.3457	0.3064	0.1390	0.1505	0.1434	0.0519	0.0677	0.0499
	(0.0036)	(0.0037)	(0.0469)	(0.0206)	(0.0074)	(0.0237)	(0.0103)	(0.0073)
1985	0.3689	0.3054	0.1474	0.1547	0.1381	0.0640	0.0610	0.0482
	(0.0076)	(0.0034)	(0.0443)	(0.0353)	(0.0091)	(0.0195)	(0.0139)	(0.0098)
1986	0.3603	0.3031	0.1453	0.1552	0.1363	0.0607	0.0657	0.0474
	(0.0049)	(0.0041)	(0.0424)	(0.0179)	(0.0085)	(0.0227)	(0.0120)	(0.0077)
1987	0.3620	0.3125	0.1479	0.1562	0.1484	0.0646	0.0626	0.0479
	(0.0047)	(0.0040)	(0.0456)	(0.0251)	(0.0109)	(0.0293)	(0.0205)	(0.0107)
1988	0.3749	0.3121	0.1500	0.1627	0.1408	0.0660	0.0620	0.0500
	(0.0077)	(0.0044)	(0.0366)	(0.0330)	(0.0133)	(0.0371)	(0.0148)	(0.0133)
1989	0.3841	0.3144	0.1526	0.1671	0.1499	0.0670	0.0738	0.0596
	(0.0072)	(0.0038)	(0.0512)	(0.0248)	(0.0107)	(0.0352)	(0.0279)	(0.0136)
1990	0.3771	0.3115	0.1506	0.1623	0.1441	0.0667	0.0705	0.0517
	(0.0055)	(0.0038)	(0.0578)	(0.0234)	(0.0117)	(0.0400)	(0.0217)	(0.0107)
1991	0.3682	0.3154	0.1618	0.1596	0.1429	0.0661	0.0722	0.0698
	(0.0038)	(0.0035)	(0.0378)	(0.0279)	(0.0154)	(0.0362)	(0.0261)	(0.0086)
1992	0.4079	0.3499	0.1668	0.1801	0.1540	0.0766	0.0743	0.0666
	(0.0053)	(0.0045)	(0.0550)	(0.0437)	(0.0131)	(0.0464)	(0.0255)	(0.0129)
1993	0.4098	0.3608	0.1714	0.1823	0.1701	0.0789	0.0840	0.0804
	(0.0044)	(0.0049)	(0.1015)	(0.0586)	(0.0244)	(0.0801)	(0.0484)	(0.0211)
1994	0.4774	0.4035	0.1889	0.1833	0.2020	0.0935	0.0986	0.0858
	(0.0067)	(0.0057)	(0.1507)	(0.1213)	(0.0472)	(0.1030)	(0.0852)	(0.0272)
1995	0.4645	0.3947	0.1872	0.2078	0.1499	0.0870	0.0961	0.0879
	(0.0060)	(0.0052)	(0.1150)	(0.0881)	(0.0383)	(0.0824)	(0.0511)	(0.0234)
1996	0.4579	0.3951	0.1872	0.1992	0.2012	0.0872	0.0996	0.0811
	(0.0065)	(0.0054)	(0.1249)	(0.0857)	(0.0481)	(0.0842)	(0.0720)	(0.0210)
1997	0.4601	0.4079	0.1900	0.2020	0.1920	0.0878	0.0891	0.0797
	(0.0049)	(0.0062)	(0.1235)	(0.0820)	(0.0473)	(0.1002)	(0.0811)	(0.0263)

Table 2: Inequality and Polarization Indices for the US 1984-1997 (Standard errors in parenthesis; obtained using 200 bootstrap repetitions)

Germany	% Horizontal Demarcation Line						
Year	0	2.5	5	7.5	10	12.5	15
1984	0.539 (4802.32)	0.167 (1931.38)	0.061 (1931.38)	0.056 (1931.38)	**0.031** (1931.38)	0.550 (1271.07)	0.558 (1279.07)
1985	0.668 (3606.93)	0.014 (2577.00)	0.006 (2577.00)	0.009 (2577.00)	0.005 (2577.00)	0.010 (2577.00)	**0.008** (2577.00)
1986	0.135 (5595.99)	0.103 (2288.30)	0.092 (2288.30)	**0.089** (2288.30)	0.106 (2185.76)	0.124 (2185.76)	0.134 (2140.10)
1987	0.621 (3329.09)	**0.404** (1869.58)	0.316 (1839.83)	0.096 (1839.83)	0.868 (1143.26)	0.866 (1143.26)	0.864 (1143.26)
1988	0.000 (14918.23)	**0.032** (3302.48)	0.241 (1657.96)	0.191 (1657.96)	0.939 (935.84)	0.924 (935.84)	0.915 (935.84)
1989	0.086 (8618.43)	**0.275** (2094.23)	0.173 (1947.44)	0.116 (1874.05)	0.243 (1553.69)	0.237 (1553.69)	0.230 (1553.69)
1990	0.005 (23175.85)	0.001 (8245.05)	0.002 (8245.05)	0.007 (8245.05)	0.004 (8245.05)	0.002 (8245.05)	**0.003** (8245.05)
1991	0.010 (17601.69)	**0.026** (3376.51)	0.079 (1677.85)	0.381 (1262.18)	0.357 (1262.18)	0.339 (1262.18)	0.330 (1262.18)
1992	0.334 (5964.32)	**0.006** (3851.74)	0.115 (1834.37)	0.236 (1654.85)	0.623 (1394.00)	0.554 (1394.00)	0.578 (1394.00)
1993	0.011 (22050.59)	0.265 (2762.72)	**0.001** (2762.72)	0.008 (1964.15)	0.026 (1678.50)	0.018 (1678.50)	0.013 (1678.50)
1994	0.131 (9326.62)	**0.175** (2439.03)	0.135 (1810.67)	0.034 (1810.67)	0.014 (1810.67)	0.017 (1707.79)	0.024 (1707.79)
1995	0.225 (12686.90)	**0.199** (2064.50)	0.135 (2064.50)	0.043 (2064.50)	0.065 (1484.09)	0.033 (1484.09)	0.032 (1484.09)
1996	0.972 (2819.25)	0.684 (1555.67)	0.519 (1555.67)	0.323 (1555.67)	0.136 (1555.67)	0.126 (1555.67)	**0.048** (1555.67)
1997	0.073 (6091.27)	0.248 (2301.52)	0.171 (2301.52)	0.009 (2301.52)	**0.000** (2301.52)	0.002 (1980.17)	0.801 (887.18)
1998	0.398 (5360.60)	**0.246** (2362.40)	0.243 (1921.27)	0.132 (1668.13)	0.035 (1668.13)	0.032 (1668.13)	0.262 (1196.02)
1999	0.470 (4321.58)	0.092 (2893.92)	**0.054** (2893.92)	0.265 (1713.28)	0.067 (1713.28)	0.116 (1608.37)	0.099 (1608.37)
2000	0.634 (12899.40)	**0.093** (3352.74)	0.608 (1732.25)	0.428 (1486.59)	0.466 (1329.42)	0.854 (956.34)	0.828 (95634)

Table 3: Unimodality Test for Germany 1984-2000 (p-value for different horizontal demarcation lines (HDL); obtained using 1000 bootstrap repetitions; critical bandwidth in parenthesis; relevant p-values in bold)

US	% Horizontal Demarcation Line						
Year	0	2.5	5	7.5	10	12.5	15
1984	0.762 (4698.94)	0.047 (1763.76)	0.031 (1763.76)	0.025 (1763.76)	0.012 (1763.76)	0.009 (1763.76)	**0.007** (1763.76)
1985	0.473 (7176.40)	**0.001** (2558.94)	0.000 (2249.18)	0.000 (2249.18)	0.000 (2249.18)	0.000 (2249.18)	0.001 (2249.18)
1986	0.060 (10628.01)	**0.203** (1992.84)	0.263 (1772.27)	0.266 (1772.27)	0.210 (1772.27)	0.209 (1772.27)	0.190 (1772.27)
1987	0.005 (11417.35)	**0.002** (2738.38)	0.241 (1512.18)	0.145 (1512.18)	0.132 (1512.18)	0.096 (1512.18)	0.088 (1512.18)
1988	0.898 (5465.21)	0.297 (1865.99)	**0.043** (1865.99)	0.072 (1675.16)	0.022 (1675.16)	0.014 (1675.16)	0.017 (1675.16)
1989	0.035 (30658.25)	**0.000** (3480.43)	0.346 (1577.85)	0.205 (1577.85)	0.416 (1313.69)	0.370 (1313.69)	0.990 (460.15)
1990	0.066 (25180.94)	**0.181** (2236.20)	0.529 (1540.56)	0.569 (1400.71)	0.432 (1400.71)	0.316 (1400.71)	0.299 (1400.71)
1991	0.000 (29230.08)	**0.397** (1694.83)	0.455 (1577.34)	0.498 (1577.34)	0.293 (1577.34)	0.236 (1577.34)	0.329 (1443.06)
1992	0.016 (28249.24)	**0.546** (2355.51)	0.801 (1981.59)	0.956 (1574.49)	0.963 (1574.49)	0.934 (1574.49)	0.935 (1574.49)
1993	0.002 (24199.36)	**0.270** (3080.38)	0.511 (2134.87)	0.855 (1668.29)	0.835 (1668.29)	0.819 (1668.29)	0.789 (1668.29)
1994	0.001 (30594.21)	**0.118** (3125.20)	0.083 (3090.91)	0.011 (3090.91)	0.007 (3090.91)	0.010 (3090.91)	0.010 (3090.91)
1995	0.073 (38647.81)	0.099 (3144.43)	**0.002** (3144.43)	0.790 (1377.69)	0.775 (1377.69)	0.759 (1377.69)	0.782 (1377.69)
1996	0.784 (13555.72)	**0.538** (2438.77)	0.578 (2274.82)	0.168 (2274.82)	0.081 (2274.82)	0.058 (2274.82)	0.072 (2274.82)
1997	0.360 (20880.42)	0.279 (3403.69)	0.071 (3403.69)	0.010 (3403.69)	**0.000** (3403.69)	0.935 (996.47)	0.979 (763.04)

Table 4: Unimodality Test for the US 1984-1997 (p-value for different horizontal demarcation lines (HDL); obtained using 1000 bootstrap repetitions; critical bandwidth in parenthesis; relevant p-values in bold)